

THE GROUPS S^3 AND $SO(3)$ HAVE NO INVARIANT BINARY k -NETWORK

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ABSTRACT. A family \mathcal{N} of closed subsets of a topological space X is called a *closed k -network* if for each open set $U \subset X$ and a compact subset $K \subset U$ there is a finite subfamily $\mathcal{F} \subset \mathcal{N}$ with $K \subset \bigcup \mathcal{F} \subset U$. A compact space X is called *supercompact* if it admits a closed k -network \mathcal{N} which is *binary* in the sense that each linked subfamily $\mathcal{L} \subset \mathcal{N}$ is centered. A closed k -network \mathcal{N} in a topological group G is *invariant* if $xAy \in \mathcal{N}$ for each $A \in \mathcal{N}$ and $x, y \in G$. According to a result of Kubiś and Turek [3], each compact (abelian) topological group admits an (invariant) binary closed k -network. In this paper we prove that the compact topological groups S^3 and $SO(3)$ admit no invariant binary closed k -network.

1. INTRODUCTION

In this note we shall discuss the problem of the existence of invariant binary k -networks for compact G -spaces and compact topological groups.

A family \mathcal{A} of subsets of a set X is called

- *linked* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$;
- *centered* if $\bigcap \mathcal{F} \neq \emptyset$ for any finite subfamily $\mathcal{F} \subset \mathcal{A}$;
- *binary* if each linked subfamily of \mathcal{F} is centered.

A family \mathcal{A} of subsets of a topological space X is called a *k -network* if for any open set $U \subset X$ and a compact subset $K \subset U$ there is a finite subfamily $\mathcal{F} \subset \mathcal{A}$ with $K \subset \bigcup \mathcal{F} \subset U$, see [2, §11]. If each set $A \in \mathcal{A}$ of a k -network is closed in X , then \mathcal{A} will be called a *closed k -network*.

A compact space X is called *supercompact* if X admits a subbase of the topology such that each cover of X by elements of the subbase contains a two-element subcover, see [5]. The following useful characterization of the supercompactness can be derived from Lemma 3.1 of [3]:

Theorem 1. *A compact Hausdorff space X is supercompact if and only if X admits a binary closed k -network.*

In [4] C.Mills proved that each compact topological group G is supercompact, that is G admits a binary closed k -network \mathcal{N} . This result was reproved by W.Kubiś and S.Turek [3] who observed that for an abelian compact topological group G one can construct \mathcal{N} so that it is *left-invariant* in the sense that $xA \in \mathcal{N}$ for each $A \in \mathcal{N}$ and $x \in G$. They also asked if such a left-invariant binary k -network can be constructed in each compact topological group.

1991 *Mathematics Subject Classification.* 54D30; 22C05.

Key words and phrases. Compact topological group, supercompact space, invariant binary k -network.

It is natural to consider this problem in the more general context of G -spaces. By a G -space we understand a topological space X endowed with a continuous action $\alpha : G \times X \rightarrow X$ of a topological group G . A family \mathcal{F} of subsets of a G -space X will be called G -invariant if $gF \in \mathcal{F}$ for each $F \in \mathcal{F}$ and each $g \in G$.

A compact G -space X will be called G -supercompact if X admits a G -invariant binary closed k -network.

Problem 1. *Which compact G -spaces are G -supercompact?*

We shall resolve this problem for the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ in the Euclidean space \mathbb{R}^{n+1} , endowed with the natural action of the group $\mathrm{SO}(n+1)$ (of orientation preserving linear isometries of \mathbb{R}^{n+1}).

Example 1. (1) *The 0-sphere $S^0 = \{-1, 1\}$ in \mathbb{R} is $\mathrm{SO}(1)$ -supercompact because the family $\mathcal{F}_0 = \{\{-1\}, \{1\}\}$ of singletons in an $\mathrm{SO}(1)$ -invariant binary closed k -network for S^0 .*
 (2) *The 1-sphere S^1 is $\mathrm{SO}(2)$ -supercompact because the family \mathcal{F}_1 of all closed connected subsets of diameter less than $\sqrt{3}$ in S^1 is an $\mathrm{SO}(2)$ -invariant binary closed k -network for the circle S^1 .*

It turns out that S^0 and S^1 are the unique examples of $\mathrm{SO}(n+1)$ -supercompact spheres S^n .

Theorem 2. *The unit sphere S^n in the Euclidean space \mathbb{R}^{n+1} is $\mathrm{SO}(n+1)$ -supercompact if and only if $n \leq 1$.*

This theorem will be proved in Section 2. Now we shall apply this theorem for finding an example of a compact topological group that admits no invariant binary closed k -network.

A family \mathcal{F} of subsets of a group G will be called

- *left-invariant* (resp. *right-invariant*) if for each $F \in \mathcal{F}$ and $g \in G$ we get $gF \in \mathcal{F}$ (resp. $Fg \in \mathcal{F}$);
- *invariant* if \mathcal{F} is both left-invariant and right-invariant.

It is well-known that the 3-dimensional sphere S^3 has the structure of a compact topological group. Namely, S^3 is a group with respect to the operation of multiplication of quaternions (with unit norm). It is known [1, §4.1] that for each isometry $f \in \mathrm{SO}(4)$ of S^3 there are quaternions $a, b \in S^3$ such that $f(x) = axb$ for all $x \in S^3$. This implies that a family \mathcal{F} of subsets of the group S^3 is invariant if and only if it is $\mathrm{SO}(4)$ -invariant. Now we see that Theorem 2 implies:

Corollary 1. *The compact topological group S^3 admits no invariant binary closed k -network.*

It is known that the quotient group $S^3/\{-1, 1\}$ of S^3 by the two-element subgroup $\{-1, 1\}$ is isomorphic to the special orthogonal group $\mathrm{SO}(3)$. Using this fact, we can deduce from Corollary 1 the following:

Corollary 2. *The compact topological group $\mathrm{SO}(3)$ admits no invariant binary closed k -network.*

Problem 2. *Has the group S^3 or $\mathrm{SO}(3)$ a left-invariant binary k -network?*

Problem 3. *Let G be a compact abelian group and X is a compact metrizable G -space. Is X G -supercompact?*

Problem 4. *Let G be a metrizable (separable) abelian topological group. Has G an invariant binary closed k -network?*

2. PROOF OF THEOREM 2

First we fix some notation. By $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$ we denote the standard inner product of the Euclidean space \mathbb{R}^n . This inner product generates the norm $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle$. By $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ we shall denote the unit sphere in \mathbb{R}^{n+1} .

For an Euclidean space $E = \mathbb{R}^n$ let E^* be the dual space of E , i.e., the space of linear functionals on E endowed with the sup-norm. By Riesz's Representation Theorem, for each functional $y^* \in E^*$ there is a unique vector $y \in E$ such that $y^*(x) = \langle y, x \rangle$. So we can identify E^* with E .

A *convex body* in an Euclidean space E is a convex subset $C \subset E$ with non-empty interior in E . By ∂C we denote the boundary of C in E .

A functional $y^* \in E^*$ will be called a *support functional* to C at a point $c \in \partial C$ if

$$y^*(c) = \max x^*(C) > \inf y^*(C).$$

By the Hahn-Banach Theorem, each point $c \in \partial C$ of a convex body $C \subset E$ has a support functional y^* with unit norm. If such a support functional is unique, then c is called a *smooth point* of ∂C . It follows from the classical Mazur's Theorem on the differentiability of continuous convex functions on E that the set of smooth points is dense in ∂C .

In an obvious way Theorem 2 follows from Example 1 and the following theorem:

Theorem 3. *For any $n \geq 2$ and any closed subset $A \subset S^n$ of diameter $0 < \text{diam}(A) \leq 1$ there is an isometry $f \in SO(n+1)$ such that the family $\{A, f(A), f^2(A)\}$ is linked but not centered.*

Proof. Let $E = \mathbb{R}^{n+1}$ and E^* be the dual space to E . By S^* we denote the unit sphere in E^* .

Lemma 1. *There are distinct points $a_0, a_1 \in A$ and a vector $b \in S^*$ such that $\langle b, a_0 \rangle = 0 = \max_{a \in A} \langle b, a \rangle$ and $\langle b, a_1 \rangle > -\frac{1}{2}\|a_1 - a_0\|$.*

Proof. The lemma trivially holds if there are a vector $b \in S^*$ and two distinct points $a_0, a_1 \in A$ such that $\langle b, a_0 \rangle = \langle b, a_1 \rangle = \max_{a \in A} \langle b, a \rangle = 0$.

So, assume that no such vectors b, a_0, a_1 exist. Let L_A be the linear hull of the set A and $C \subset L_A$ be the closed convex hull of the set $A \cup \{0\}$ in L_A . Since the set $A \subset S^n$ contains more than one point, the linear space L_A has dimension $\dim L_A \geq 2$. It is clear that C is a convex body in L_A . By Mazur's Theorem, the set of smooth points is a dense in the boundary ∂C . Consequently, there is a smooth point $c \in \partial C$ such that $0 < \|c\| < 1$. Let $b^* \in L_A^*$ be the unique norm one support functional to C at the point c . Let $a_0 = \frac{c}{\|c\|}$ and observe that $a_0 \in \text{conv}(A) \subset C$.

Since b^* is a support functional at c , we get $b^*(c) = \max b^*(C) \geq b^*(0) = 0$. We claim that $b^*(c) = 0$. The strict inequality $b^*(c) > 0$ would imply $b^*(c) = 0 = \max b^*(C) \geq b^*(a_0) = \frac{b^*(c)}{\|c\|}$ and $\|c\| \geq 1$, which contradicts the choice of c .

Let us show that the point $a_0 = c/\|c\|$ belongs to the set A . Since $c \in \text{conv}(A \cup \{0\}) \setminus \{0\}$ by the Caratheodory Theorem, there are pairwise distinct

points $a_1, \dots, a_k \in A$ and positive real numbers $\lambda_1, \dots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i \leq 1$ and $c = \sum_{i=1}^k \lambda_i a_i$. This equality and $b^*(c) = 0 = \max b^*(A)$ imply that $b^*(a_i) = 0$ for all $1 \leq i \leq k$. Now our assumption guarantees that $k = 1$ (otherwise, a_1 and a_2 are two distinct points with $b^*(a_1) = b^*(a_2) = \max b^*(A) = 0$, which is forbidden by our assumption). Therefore, $c = \lambda_1 a_1$ and hence $a_0 = c/\|c\| = c/\lambda_1 = a_1 \in A$.

Let $c^* \in L_A^*$ be any functional with unit norm such that $c^*(a_0) = 0$ and $0 < \|b^* - c^*\| \leq \frac{1}{2}$. Since the functional $c^* \neq b^*$ is not support at the point c , there is a point $a_1 \in A$ such that $c^*(a_1) > 0$.

Observe that

$$\begin{aligned} b^*(a_1) &= b^*(a_1 - a_0) \geq c^*(a_1 - a_0) - \|c^* - b^*\| \cdot \|a_1 - a_0\| = \\ &= c^*(a_1) - \frac{1}{2}\|a_1 - a_0\| > -\frac{1}{2}\|a_1 - a_0\| \end{aligned}$$

By Riesz's Representation Theorem, the functional b^* can be identified with a unique vector $b \in L_A \subset E$ such that $b^*(x) = \langle b, x \rangle$ for all $x \in L$. The vector b and the points a_0, a_1 have the properties required in Lemma 1. \square

Let L be the 3-dimensional linear subspace of E generated by the vectors b, a_0, a_1 (from Lemma 1) and let $L^\perp \subset E$ be its orthogonal complement. Then the space E decomposes into the direct sum $L \oplus L^\perp$.

Find a (unique) point a_2 in the 2-sphere $L \cap S^n$ such that $\|a_2 - a_0\| = \|a_2 - a_1\| = \|a_1 - a_0\|$ and $\langle b, a_2 \rangle > 0$. Let $c = \frac{1}{3}(a_0 + a_1 + a_2)$ be the center of the equilateral triangle $\triangle a_0, a_1, a_2$. It follows from $\langle b, a_0 \rangle = 0$ and $0 \geq \langle b, a_1 \rangle > -\frac{1}{2}\|a_0 - a_1\|$ that $\langle b, a_2 \rangle > \frac{1}{2}\|a_0 - a_1\|$. Consequently,

$$(*) \quad \langle b, c \rangle = \frac{1}{3}(\langle b, a_1 \rangle + \langle b, a_2 \rangle) > 0.$$

Claim 1. $\langle c, a \rangle > 0$ for each $a \in A$.

Proof. Observe that $\langle a_2, a_0 \rangle = \frac{1}{2}(\|a_2\|^2 + \|a_0\|^2 - \|a_2 - a_0\|^2) \geq \frac{1}{2}(1 + 1 - 1) = \frac{1}{2}$ and then $\| -a_2 - a_0 \|^2 = \|a_2\|^2 + \|a_0\|^2 + 2\langle a_2, a_0 \rangle \geq 3$, which implies that $-a_2 \notin A$ because $\text{diam}(A) \leq 1$. Then for each $a \in A$ we get $a_2 \neq -a$ and hence $\langle a_2, a \rangle > -\|a_2\| \cdot \|a\| = -1$.

On the other hand, for $i \in \{0, 1\}$ we get

$$\langle a_i, a \rangle = \frac{1}{2}(\|a_i\|^2 + \|a\|^2 - \|a_0 - a\|^2) \geq \frac{1}{2}(1 + 1 - 1) = \frac{1}{2}.$$

Then

$$\langle c, a \rangle = \frac{1}{3}\langle a_0 + a_1 + a_2, a \rangle = \frac{1}{3}(\langle a_0, a \rangle + \langle a_1, a \rangle + \langle a_2, a \rangle) > \frac{1}{3}(\frac{1}{2} + \frac{1}{2} - 1) = 0.$$

\square

Let $R : L \rightarrow L$ be the rotation of the 3-dimensional Euclidean space L around the axis $\mathbb{R}c$ on the angle $2\pi/3$ such that $R(a_0) = a_1$, $R(a_1) = a_2$ and $R(a_2) = a_0$. Extend R to an isometry $f \in \text{SO}(n+1)$ of $E = L \oplus L^\perp$ letting $f(x+y) = R(x) + y$ for $(x, y) \in L \times L^\perp$. It remains to prove:

Claim 2. The system $\mathcal{L} = \{A, f(A), f^2(A)\}$ is linked but not centered.

Proof. The linkedness of the system \mathcal{L} follows from the inclusion $\{a_0, a_1\} \subset A$ and the linkedness of the system

$$\{\{a_0, a_1\}, \{a_1, a_2\}, \{a_2, a_0\}\} = \{\{a_0, a_1\}, f(\{a_0, a_1\}), f^2(\{a_0, a_1\})\}.$$

To see that \mathcal{L} is not centered, consider the half-spaces $H_b = \{x \in E : \langle b, x \rangle \leq 0\}$ and $H_c = \{x \in E : \langle c, x \rangle > 0\}$. The choice of the vectors b, a_0, a_1, a_2 guarantees that $a_0, a_1 \in A \subset H_b$ but $a_2, c \notin H_b$. By Claim 1, $A \subset H_c$.

Let $H_c^L = H_c \cap L$ and $H_b^L = H_b \cap L$. The inclusions $b, c \in L$ imply that $H_b = H_b^L \oplus L^\perp$ and $H_c = H_c^L \oplus L^\perp$.

It follows that $R(H_c^L) = H_c^L$ and hence $f(H_c) = H_c$. Observe that

$$A \cap f(A) \cap f^2(A) \subset H_c \cap H_b \cap f(H_b) \cap f^2(H_b) = (H_c^L \cap H_b^L \cap R(H_b^L) \cap R^2(H_b^L)) \oplus L^\perp.$$

Now to see that $A \cap f(A) \cap f^2(A) = \emptyset$ it suffices to prove that the intersection $H^L = H_c^L \cap H_b^L \cap R(H_b^L) \cap R^2(H_b^L)$ is empty. Assuming that this intersection contains some point h , we conclude that it contains its rotations $R(h)$ and $R^2(h)$ and also the center $c_h = \frac{1}{3}(h + R(h) + R^2(h))$ of the equilateral triangle $\{h, R(h), R^2(h)\}$ (by the convexity of H^L). The center c_h lies on the axis $\mathbb{R} \cdot c$ of the rotation R . Taking into account that $c_h \in H_c$, we conclude that $\langle c, c_h \rangle > 0$ and hence $c \in (0, +\infty) \cdot c_h \subset H_b$, which contradicts the inequality (*). \square

\square

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